Equations of elastic wave propagation and magnetoacoustics

By Pierre-Yves Guerder

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Description: Description of the main equations of elastic wave propagation and magnetoacoustics.

Introduction

This document describes the equations of elastic wave propagation and magnetoacoustics as presented by Bou Matar *et al.* [1, 2].

Elastic wave propagation

We write v_i the components of the particle velocity vector, u_i the components of the displacement vector, a_i the components of the Lagrangian position vector, ρ_0 the density, P_{ij} the components of the first Piola-Kirchhoff stress tensor, t the time, W the elastic energy density and F the deformation gradient.

The equation of motion for 2D plane strain elastic waves in heterogeneous solid medium, using Einstein's summation convention, is:

$$\rho_0 \frac{\partial v_i}{\partial t} = \frac{\partial P_{ij}}{\partial a_j} \tag{1}$$

It is completed by the Hooke's law:

$$P_{ij} = C_{ijkl} \frac{\partial u_k}{\partial a_i} = \rho_0 \frac{\partial W}{\partial F_{ij}} \tag{2}$$

The relation between the particle velocity and the deformation gradient is

$$\frac{\partial F_{ij}}{\partial t} = \frac{\partial v_i}{\partial a_j} \text{ or } F_{ij} = \frac{\partial u_i}{\partial a_j} \tag{3}$$

Conservative form

The elastic wave equations can be written as a conservation equation which links the time derivative of a state vector $\mathbf{Q}$ to the space derivatives of the flux vectors $\mathbf{F}$ and $\mathbf{G}$ and to the source vector $\mathbf{S}$:

$$\frac{\partial \mathbf{Q}(t,\mathbf{x})}{\partial t} = \frac{\partial \mathbf{F}(t,\mathbf{x})}{\partial x} + \frac{\partial \mathbf{G}(t,\mathbf{x})}{\partial y} + \partial \mathbf{S}(t,\mathbf{x}) \tag{4}$$

With $\mathbf{x} = [x, y]$ the direction vector and:

$$\mathbf{Q} = \begin{pmatrix} \rho_0 v_1 \\ \rho_0 v_2 \\ F_{11} \\ F_{22} \\ F_{12} \\ F_{21} \end{pmatrix}, \ \mathbf{F} = \begin{pmatrix} P_{11} \\ P_{21} \\ v_1 \\ 0 \\ 0 \\ v_2 \end{pmatrix}, \ \mathbf{G} = \begin{pmatrix} P_{12} \\ P_{22} \\ 0 \\ v_2 \\ v_1 \\ 0 \end{pmatrix} \tag{5}$$

Magnetoacoustic

We write ρ_0 the density before deformation, ρ the density after deformation, μ_0 the permeability of vacuum, γ the gyromagnetic ratio (ratio of the magnetic dipole moment μ' to its angular momentum $L = mvr$), x_i the Eulerian coordinates, $\mathbf{H}$ the Maxwellian magnetic field and $\mathbf{H}_{\text{eff}}$ the effective internal magnetic field.

In the case of a ferromagnet magnetized to saturation, the amplitude of the magnetization per unit mass μ is a constant μ_s. We can also write that the magnetization per unit volume $\mathbf{M} = \rho\mu$ is a constant $\mathbf{M_s}$.

The equation of motion for 2D plane strain elastic waves in heterogeneous solid medium is modified and completed by the Landau-Lifschitz equation:

$$\rho_0 \frac{\partial v_i}{\partial t} = \frac{\partial P_{ij}}{\partial a_j} + \rho_0 \mu_n \frac{\partial H_n}{\partial a_j} \frac{\partial a_j}{\partial x_i} \text{ and } \frac{\partial \mu}{\partial t} = -\gamma \mu_{\mathbf{0}} \mu \times \mathbf{H}_{\text{eff}} \tag{6}$$

The effective internal magnetic field and the Piola-Kirchhoff stress tensor are:

$$\mathbf{H}_{\text{eff},i} = H_i - \frac{1}{\mu_0 \rho_0} \frac{\partial U}{\partial \mu_i} + \frac{1}{\mu_0 \rho_0} \frac{\partial}{\partial a_s} \frac{\partial U}{\partial(\partial \mu_i / \partial a_s)} \text{ and } P_{ij} = \frac{\partial U}{\partial \eta_{pj}} \frac{\partial x_i}{\partial a_p} \tag{7}$$

The strain tensor is:

$$\eta_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i} + \frac{\partial u_k}{\partial a_i} \frac{\partial u_k}{\partial a_j} \right) \tag{8}$$

The local energy per unit volume is the sum of the magnetocrystalline anisotropy, the magnetoelastic coupling and elastic energies:

$$U = U_{\text{an}} + U_{\text{me}} + U_e \tag{9}$$

References

[1] O. B. Matar, J. F. Robillard, J. O. Vasseur, A.-C. Hladky-Hennion, P. A. Deymier, P. Pernod, and V. Preobrazhensky, "Band gap tunability of magneto-elastic phononic crystal," *Journal of Applied Physics*, vol. 111, no. 054901, pp. 1–12, 2012. 1

[2] O. B. Matar, P.-Y. Guerder, Y. Li, B. Vandewoestyne, and K. V. D. Abeele, "A nodal discontinuous galerkin finite element method for nonlinear elastic wave propagation," *J. Acoust. Soc. Am.*, vol. 131, no. 5, pp. 3650–3663, 2012. 1